

Is Lebesgue measure the only σ -finite invariant Borel measure?

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September 27, 2011

Abstract

R. D. Mauldin asked if every translation invariant σ -finite Borel measure on \mathbb{R}^d is a constant multiple of Lebesgue measure. The aim of this paper is to show that the answer is “yes and no”, since surprisingly the answer depends on what we mean by Borel measure and by constant. We present Mauldin’s proof of what he called a folklore result, stating that if the measure is only defined for Borel sets then the answer is affirmative. Then we show that if the measure is defined on a σ -algebra *containing* the Borel sets then the answer is negative. However, if we allow the multiplicative constant to be infinity, then the answer is affirmative in this case as well. Moreover, our construction also shows that an isometry invariant σ -finite Borel measure (in the wider sense) on \mathbb{R}^d can be non- σ -finite when we restrict it to the Borel sets.

Introduction

It is classical that, up to a nonnegative multiplicative constant, Lebesgue measure is the unique locally finite translation invariant Borel measure on \mathbb{R}^d . R. D. Mauldin [7] asked if we can replace local finiteness by σ -finiteness. Then he himself gave an affirmative answer in the case when Borel measure means a measure defined on the σ -algebra of Borel sets. He referred to this theorem as folklore result, but for the sake of completeness we include his proof here. Let λ_d denote d -dimensional Lebesgue measure, and $B + t = \{b + t : b \in B\}$.

Theorem 0.1 *Let μ be a σ -finite translation invariant measure defined on the Borel subsets of \mathbb{R}^d . Then there exists $c \in [0, \infty)$ such that $\mu(B) = c\lambda_d(B)$ for every Borel set B .*

*Partially supported by Hungarian Scientific Foundation grant no. 37758, 049786 and F 43620.

†Partially supported by Hungarian Scientific Foundation grant no. 049786 and F 43620.

MSC codes: Primary 28C10; Secondary 28A05, 28A10

Key Words: Lebesgue, Borel, measure, unique, translation, isometry, invariant, σ -finite

Proof. First we prove that μ is absolutely continuous with respect to λ_d . Let $B \subset \mathbb{R}^d$ be a Borel set with $\lambda_d(B) = 0$. Define $\tilde{B} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x + y \in B\}$. This set is clearly Borel, and as both λ_d and μ are σ -finite measures, we can apply the Fubini theorem. Note that the x -section $\tilde{B}_x = \{y : (x, y) \in \tilde{B}\} = B - x$, and similarly $\tilde{B}^y = \{x : (x, y) \in \tilde{B}\} = B - y$. So by Fubini $\lambda_d(B) = 0$ implies $(\lambda_d \times \mu)(\tilde{B}) = 0$. Hence $\mu(B - x) = 0$ for λ_d -almost every x , but μ is translation invariant, so $\mu(B) = 0$.

Therefore by the Radon-Nikodým theorem there exists a Borel function $f : \mathbb{R}^d \rightarrow [0, \infty]$ such that $\mu(B) = \int_B f d\lambda_d$ for every Borel set B . Clearly

$$\mu(B) = \mu(B + t) = \int_{B+t} f d\lambda_d = \int_B f(x - t) d\lambda_d(x)$$

for every t and every Borel set B . Hence the uniqueness of the Radon-Nikodým derivative implies that for every t for Lebesgue almost every x the equation

$$f(x - t) = f(x) \tag{1}$$

holds.

In order to complete the proof it is clearly sufficient to show that there is a constant $c \in [0, \infty)$ such that $f(x) = c$ holds for λ_d -almost every x . Suppose on the contrary that there are real numbers $r_1 < r_2$ such that the Borel sets $\{x : f(x) < r_1\}$ and $\{x : f(x) > r_2\}$ are of positive Lebesgue measure. Let d_1 and d_2 be Lebesgue density points of the two sets, respectively. But then equation (1) fails for $t = d_1 - d_2$, a contradiction. \square

However, in the literature there are two different notions that are referred to as Borel measure. The first one is measures defined only for Borel sets (see e.g. [4], [8]), while the second one is measures defined on σ -algebras *containing* the Borel sets (see e.g. [1], [6]).

In the rest of the paper we investigate Mauldin's question in the case of the more general notion. As a side effect, we also show that σ -finiteness is also sensitive to the definition of Borel measure. This question is closely related to [2], and was implicitly asked there.

1 The negative result

In this section we prove somewhat more than just a negative answer to Mauldin's question.

Theorem 1.1 *There exists an isometry invariant σ -finite measure μ defined on an isometry invariant σ -algebra \mathcal{A} containing the Borel subsets of \mathbb{R}^d such that, for every Borel set B , if $\lambda_d(B) = 0$ then $\mu(B) = 0$, while if $\lambda_d(B) > 0$ then $\mu(B) = \infty$.*

Before the proof we need a lemma. $\text{Isom}(\mathbb{R}^d)$ is the group of isometries of \mathbb{R}^d , the symbol $|X|$ denotes the cardinality of a set X , the continuum cardinality is

denoted by 2^ω , Δ stands for symmetric difference of two sets, and a set $P \subset \mathbb{R}^d$ is perfect if it is nonempty, closed and has no isolated points.

Lemma 1.2 *There exists a disjoint decomposition $\mathbb{R}^d = \bigcup_{n=0}^\infty A_n$ such that $|\varphi(A_n) \Delta A_n| < 2^\omega$ for every $n \in \mathbb{N}$ and every $\varphi \in \text{Isom}(\mathbb{R}^d)$, and such that $|A_n \cap P| = 2^\omega$ for every $n \in \mathbb{N}$ and every perfect set $P \subset \mathbb{R}^d$.*

Proof. We say that a set $A \subset \mathbb{R}^d$ is $< 2^\omega$ -invariant, if $|\varphi(A) \Delta A| < 2^\omega$ for every $\varphi \in \text{Isom}(\mathbb{R}^d)$. As $\text{Isom}(\mathbb{R}^d)$ is closed under inverses, this is equivalent to $|\varphi(A) \setminus A| < 2^\omega$ for every $\varphi \in \text{Isom}(\mathbb{R}^d)$.

It is enough to construct a sequence A_n of disjoint $< 2^\omega$ -invariant sets such that $|A_n \cap P| = 2^\omega$ for every $n \in \mathbb{N}$ and every perfect set $P \subset \mathbb{R}^d$, since then clearly $\mathbb{R}^d \setminus \bigcup_{n=0}^\infty A_n$ is also $< 2^\omega$ -invariant, hence we can simply replace A_0 by $A_0 \cup (\mathbb{R}^d \setminus \bigcup_{n=0}^\infty A_n)$.

Now we construct such a sequence by transfinite induction. Let us enumerate $\text{Isom}(\mathbb{R}^d) = \{\varphi_\alpha : \alpha < 2^\omega\}$ and define G_α to be the group generated by $\{\varphi_\beta : \beta < \alpha\}$. Note that $|G_\alpha| < 2^\omega$. For $x \in \mathbb{R}^d$ let $G_\alpha(x) = \{\varphi(x) : \varphi \in G_\alpha\}$. Let us also enumerate the perfect subsets of \mathbb{R}^d as $\{P_\alpha : \alpha < 2^\omega\}$ such that each perfect set P is listed 2^ω many times.

Define $A_n^0 = \emptyset$ for every $n \in \mathbb{N}$. At step α we recursively construct a sequence $x_n \in P_\alpha$ such that for every $i < n$

$$[\bigcup_{\beta < \alpha} A_n^\beta \cup G_\alpha(x_n)] \cap [\bigcup_{\beta < \alpha} A_i^\beta \cup G_\alpha(x_i)] = \emptyset.$$

To see that such a choice of x_n is possible, note that the set of bad choices is

$$\bigcup_{\varphi \in G_\alpha} \varphi^{-1} \left(\bigcup_{m \neq n} \bigcup_{\beta < \alpha} A_m^\beta \cup \bigcup_{i=0}^{n-1} G_\alpha(x_i) \right),$$

which is of cardinality $< 2^\omega$. As every perfect set is of cardinality 2^ω , this set cannot cover P_α , so we can find an x_n with the required property and define $A_n^\alpha = \bigcup_{\beta < \alpha} A_n^\beta \cup G_\alpha(x_n)$. Finally, define $A_n = \bigcup_{\alpha < 2^\omega} A_n^\alpha$ for every n . These sets are clearly disjoint, they all intersect every perfect set in a set of cardinality 2^ω , and one can easily see that $\varphi_\alpha(A_n) \setminus A_n \subset A_n^\alpha$, so the A_n 's are $< 2^\omega$ -invariant. This completes the proof. \square

Proof. (Theorem 1.1) Let A_n be the sequence from the previous lemma. Define

$$\mathcal{A} = \{[\bigcup_{n=0}^\infty (A_n \cap B_n)] \Delta H : \forall n \ B_n \subset \mathbb{R}^d \text{ Borel}, H \subset \mathbb{R}^d, |H| < 2^\omega\}.$$

Clearly \mathcal{A} contains the Borel sets, as $B = [\bigcup_{n=0}^\infty (A_n \cap B)] \Delta \emptyset$.

In order to check that \mathcal{A} is closed under complements note that $(X \Delta H)^C = X^C \Delta H$, and therefore $([\bigcup_{n=0}^\infty (A_n \cap B_n)] \Delta H)^C = [\bigcup_{n=0}^\infty (A_n \cap B_n)]^C \Delta H = [\bigcup_{n=0}^\infty (A_n \cap B_n^C)] \Delta H$.

In order to show that \mathcal{A} is closed under countable unions, we need to show $\bigcup_{k=0}^\infty (X^k \Delta H^k) \in \mathcal{A}$, where

$$X^k = \bigcup_{n=0}^\infty (A_n \cap B_n^k). \quad (2)$$

Using the identity

$$Z = W \Delta W \Delta Z \quad (3)$$

(note that Δ is associative) we obtain

$$\cup_{k=0}^{\infty} (X^k \Delta H^k) = [\cup_{k=0}^{\infty} X^k] \Delta [\cup_{k=0}^{\infty} X^k] \Delta [\cup_{k=0}^{\infty} (X^k \Delta H^k)] = [\cup_{k=0}^{\infty} X^k] \Delta Y, \quad (4)$$

where $Y = [\cup_{k=0}^{\infty} X^k] \Delta [\cup_{k=0}^{\infty} (X^k \Delta H^k)]$. As

$$\cup_{k=0}^{\infty} X^k = \cup_{n=0}^{\infty} (A_n \cap (\cup_{k=0}^{\infty} B_n^k)) \quad (5)$$

it is sufficient to check that $|Y| < 2^\omega$, but this is clear, since $Y = [\cup_{k=0}^{\infty} X^k] \Delta [\cup_{k=0}^{\infty} (X^k \Delta H^k)] \subset \cup_{k=0}^{\infty} H^k$, which is of cardinality $< 2^\omega$, as a countable union of sets of cardinality $< 2^\omega$ is itself of cardinality $< 2^\omega$ (see e.g. [5, Cor. I.10.41]).

To show that \mathcal{A} is isometry invariant note that

$$\varphi([\cup_{n=0}^{\infty} (A_n \cap B_n)] \Delta H) = [\cup_{n=0}^{\infty} (\varphi(A_n) \cap \varphi(B_n))] \Delta \varphi(H). \quad (6)$$

Set

$$X = \cup_{n=0}^{\infty} (\varphi(A_n) \cap \varphi(B_n)) \text{ and } Y = \cup_{n=0}^{\infty} (A_n \cap \varphi(B_n)). \quad (7)$$

We need to show that $X \Delta \varphi(H) \in \mathcal{A}$. Using (3) again, write

$$X \Delta \varphi(H) = [Y \Delta Y \Delta X] \Delta \varphi(H) = Y \Delta [(Y \Delta X) \Delta \varphi(H)], \quad (8)$$

where we use again the associativity of Δ . Hence it is enough to show that $|(Y \Delta X) \Delta \varphi(H)| < 2^\omega$, which follows from $|H| < 2^\omega$ and $Y \Delta X = [\cup_{n=0}^{\infty} (A_n \cap \varphi(B_n))] \Delta [\cup_{n=0}^{\infty} (\varphi(A_n) \cap \varphi(B_n))] \subset \cup_{n=0}^{\infty} (A_n \Delta \varphi(A_n))$ and the $< 2^\omega$ -invariance of A_n .

Let us now define

$$\mu([\cup_{n=0}^{\infty} (A_n \cap B_n)] \Delta H) = \sum_{n=0}^{\infty} \lambda_d(B_n).$$

First we have to show that μ is well-defined. Let $[\cup_{n=0}^{\infty} (A_n \cap B_n)] \Delta H = [\cup_{n=0}^{\infty} (A_n \cap B'_n)] \Delta H'$. We claim that $\lambda_d(B_n) = \lambda_d(B'_n)$ for every n . Otherwise, without loss of generality $\lambda_d(B_n) < \lambda_d(B'_n)$, hence $B'_n \setminus B_n$ contains a perfect set P (even of positive measure). But $|P \cap A_n| = 2^\omega$ and $|H \cup H'| < 2^\omega$, hence there exists an $x \in (P \cap A_n) \setminus (H \cup H')$, and then $x \in [\cup_{n=0}^{\infty} (A_n \cap B'_n)] \Delta H'$ but $x \notin [\cup_{n=0}^{\infty} (A_n \cap B_n)] \Delta H$, a contradiction. (Recall that the A_n 's are disjoint.)

In order to prove that μ is σ -additive, let

$$\cup_{k=0}^{\infty} (X^k \Delta H^k) \quad (9)$$

be a disjoint union, where X^k is as in (2). First we claim that for every n and every $k \neq k'$ we have $\lambda_d(B_n^k \cap B_n^{k'}) = 0$. Otherwise, there exists a perfect set $P \subset B_n^k \cap B_n^{k'}$, and we can find $x \in (P \cap A_n) \setminus (H^k \cup H^{k'})$, hence $x \in$

$[\cup_{n=0}^{\infty} (A_n \cap B_n^k)] \Delta H^k$ and $x \in [\cup_{n=0}^{\infty} (A_n \cap B_n^{k'})] \Delta H^{k'}$, but then the union (9) is not disjoint, a contradiction. Therefore $\lambda_d(\cup_{k=0}^{\infty} B_n^k) = \sum_{k=0}^{\infty} \lambda_d(B_n^k)$ for every n , so by (4) and (5) we obtain $\mu(\cup_{k=0}^{\infty} (X^k \Delta H^k)) = \sum_{n=0}^{\infty} \lambda_d(\cup_{k=0}^{\infty} B_n^k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \lambda_d(B_n^k) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \lambda_d(B_n^k) = \sum_{k=0}^{\infty} \mu(X^k \Delta H^k)$.

Now we show that μ is isometry invariant. By (6), (7) and (8) we obtain that $\mu(\varphi([\cup_{n=0}^{\infty} (A_n \cap B_n)] \Delta H)) = \sum_{n=0}^{\infty} \lambda_d(\varphi(B_n))$, which clearly equals $\sum_{n=0}^{\infty} \lambda_d(B_n)$, which is $\mu([\cup_{n=0}^{\infty} (A_n \cap B_n)] \Delta H)$ by definition.

The fact that μ is σ -finite follows from $\mathbb{R}^d = \cup_{n=0}^{\infty} \cup_{K=0}^{\infty} (A_n \cap [-K, K]^d)$, since $\mu(A_n \cap [-K, K]^d) = \lambda_d([-K, K]^d) = (2K)^d < \infty$ for every n and K .

Finally, for a Borel set B we have $\mu(B) = \mu(\cup_{n=0}^{\infty} (A_n \cap B)) = \sum_{n=0}^{\infty} \lambda_d(B)$, which is zero if $\lambda_d(B) = 0$ and ∞ otherwise. \square

As an immediate corollary we obtain the following.

Corollary 1.3 *There exists an isometry invariant σ -finite measure μ defined on an isometry invariant σ -algebra \mathcal{A} containing the Borel subsets of \mathbb{R}^d such that μ restricted to the Borel sets is not equal to $c\lambda_d$ for every $c \in [0, \infty)$.*

As \mathbb{R}^d is not the union of countably many Lebesgue nullsets, the next statement is also a corollary to Theorem 1.1.

Corollary 1.4 *There exists an isometry invariant σ -finite measure μ defined on an isometry invariant σ -algebra \mathcal{A} containing the Borel subsets of \mathbb{R}^d such that μ restricted to the Borel sets is not σ -finite.*

2 The positive result

The measure μ constructed in the previous section behaves simply on Borel sets; if $\lambda_d(B) = 0$ then $\mu(B) = 0$, while if $\lambda_d(B) > 0$ then $\mu(B) = \infty$. So we can say that $\mu(B) = \infty \lambda_d(B)$ for every Borel set B . The next theorem shows that this is the only possibility.

Theorem 2.1 *Let μ be a σ -finite translation invariant measure defined on a translation invariant σ -algebra containing the Borel subsets of \mathbb{R}^d . Then there exists $c \in [0, \infty]$ such that $\mu(B) = c\lambda_d(B)$ for every Borel set B .*

Proof. If μ restricted to the Borel sets is σ -finite, then we are done by Theorem 0.1. So we can assume that this is not the case.

Lemma 2.2 *Let μ be a σ -finite translation invariant measure defined on a translation invariant σ -algebra containing the Borel subsets of \mathbb{R}^d , and suppose that μ restricted to the Borel sets is not σ -finite. Then for every Borel set B we have either $\mu(B) = 0$ or $\mu(B) = \infty$.*

Proof. Let \mathcal{B} be a maximal disjoint family of Borel sets of positive finite μ -measure. As μ is σ -finite (on \mathcal{A}), \mathcal{B} is countable, hence $B_0 = \bigcup \mathcal{B}$ is a Borel set. Define

$$\mu'(B) = \mu(B_0 \cap B) \text{ for every Borel set } B.$$

Note that this measure is only defined for Borel sets. As μ' is clearly σ -finite, we can apply the Fubini theorem for $\mu' \times \mu$ and the Borel set $\widetilde{B_0^C}$. (Using notations as in the proof of Theorem 0.1.) On one hand, $(\mu' \times \mu)(\widetilde{B_0^C}) = \int_{y \in \mathbb{R}^d} \mu'(B_0^C - y) d\mu(y) = \int_{y \in \mathbb{R}^d} \mu(B_0 \cap (B_0^C - y)) d\mu(y)$. We claim that $\mu(B_0 \cap (B_0^C - y)) = 0$ for every y , hence $(\mu' \times \mu)(\widetilde{B_0^C}) = 0$. Indeed, otherwise there is a Borel set $B \in \mathcal{B}$ such that $0 < \mu(B \cap (B_0^C - y)) < \infty$. But then for $D = B \cap (B_0^C - y)$ we obtain that the Borel set $D + y$ is disjoint from B_0 , hence from all elements of \mathcal{B} , and is of positive and finite μ -measure (since μ is translation invariant), contradicting the maximality of \mathcal{B} .

On the other hand, $0 = (\mu' \times \mu)(\widetilde{B_0^C}) = \int_{x \in \mathbb{R}^d} \mu(B_0^C - x) d\mu'(x)$. As μ restricted to the Borel sets is not σ -finite, $\mu(B_0^C - x) = \mu(B_0^C) = \infty$ for every x . Therefore we obtain $0 = \mu'(\mathbb{R}^d) = \mu(B_0)$, so $\mathcal{B} = \emptyset$ and we are done. \square

Now the proof of Theorem 2.1 will be completed by the following lemma.

Lemma 2.3 *Let μ_1 and μ_2 be σ -finite translation invariant measures defined on the (not necessarily equal) translation invariant σ -algebras \mathcal{A}_1 and \mathcal{A}_2 containing the Borel subsets of \mathbb{R}^d , and suppose that $\mu_1(\mathbb{R}^d), \mu_2(\mathbb{R}^d) > 0$. Then for every Borel set B , $\mu_1(B) = 0$ iff $\mu_2(B) = 0$.*

Proof. Apply Fubini to $\mu_1 \times \mu_2$ and \widetilde{B} . \square

Applying this lemma with $\mu_1 = \mu$ and $\mu_2 = \lambda_d$ the theorem follows. \square

From this theorem one easily obtains the following.

Corollary 2.4 *Let μ be as in the above theorem. Then μ restricted to the Borel sets is σ -finite if and only if the constant c is finite.*

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